Number and Placement of Control System Components Considering Possible Failures

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One of the first questions facing the designer of the control system for a large space structure is how many components—actuators and sensors—to specify and where to place them on the structure. This paper presents a methodology which is intended to assist the designer in making these choices. A measure of controllability is defined which is a quantitative indication of how well the system can be controlled with a given set of actuators. Similarly, a measure of observability is defined which is a quantitative indication of how well the system can be observed with a given set of sensors. Then the effect of component unreliability is introduced by computing the average expected degree of controllability (observability) over the operating lifetime of the system accounting for the likelihood of various combinations of component failures. The problem of component location is resolved by optimizing this performance measure over the admissible set of locations. The variation of this optimized performance measure with number of actuators (sensors) is helpful in deciding how many components to use.

Introduction

THE aerospace community is anticipating that future large assemblies in space will require active control not only for stationkeeping and attitude control but also for damping structure vibrations and figure control. In order that the control system can damp the many vibrational modes of such a large structure adequately, and control its figure to a specified tolerance, many sensors and actuators will be required. The system designer likely will have considerable freedom of choice as to the number of these components to specify and where to place them on the structure.

This paper presents a methodology intended to assist the designer in this choice in the early stages of system design—before a control system has been designed in detail. The usual control system design problem is to decide how the actuator commands are to be related to the sensor outputs. But this process presumes a set of sensors and actuators to be given. We address here a step which must precede this process —to decide, at least tentatively, how many sensors and actuators to incorporate in the system and where to locate them. After a proposed control system has been designed, it must, of course, be evaluated in careful detail to see if it will meet the mission requirements. That evaluation may shed additional light on the adequacy of the set of components incorporated in the design.

One factor which must be accounted for, both in the early assessment of component number and location and in the later evaluation of a specific system configuration, is the likelihood of some failures among the sensors and actuators. With the large number of components involved and the long interval desired between visits for maintenance and resupply, it would be totally unrealistic to design the control system under the assumption that all components will function properly over that interval. For example, in a control system that utilizes a total of 400 sensors and actuators—each with an exponential distribution of time to failure with a mean time to failure of 100,000 h (optimistic by today's standards) about one failure every 10 days will occur on average.

In this work a methodology for measuring the performance of a system which reflects the type, number, and placement of the sensors and actuators on the structure is developed. The measure also reflects the expected loss of performance due to component failures. This performance measure is intended to be especially useful as a guide to the choice of component number and placement in the earliest phase of system design.

Problem Definition

Modern control theory does not provide a quantitative measure of "controllability" or "observability." Controllability is simply a binary concept—either a system is controllable or it is not. A vibratory mode of a beam, for example, is not controllable by a force actuator placed exactly at one of the nodes, but it is controllable by an actuator placed just off the node. What we require is a quantitative indication of how well a system can be controlled by a given set of actuators—a fundamental measure which does not depend on the design of a control system. Similarly, a quantitative measure of how well a system can be observed by a given set of sensors is needed.

Previous Work

A number of writers have dealt with the subject of controllability and observability, three of which are somewhat related to our approach. Examples of work on this subject which are distinctly different include Horner,[1] who has considered optimum actuator placement but does it for the specific case of passive damping of a free-free beam, and Hughes and Skelton,[2] who define measures in terms of controllability and observability "norms" which apply to the individual modes of a system rather than to the system as a whole.

Juang and Rodriguez[3] formulate the linear quadratic regulator problem with an infinite time horizon. The solution defines the optimal cost as an explicit function of the initial state, and indirectly as a function of the number and location of the actuators and sensors. The expectation is taken over a defined distribution of initial conditions, producing the expected cost as a function of the actuator and sensor set only. While this approach has some appeal, our main objections to the method are: first, if there is a particular direction in the state space in which the state is not very observable or controllable, this fact is largely lost when the cost is averaged over the distribution of initial states, and, second, expected cost depends on both the actuator set and the sensor set, whereas it may be more useful to have separate measures of observability and controllability.

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Viswanathan et al.[4] developed a more sophisticated technique to be used in the case of bounded control effort. They define a "recovery region" to be the region of initial states which can be returned to the origin in a fixed time with bounds on the control variables. The measure of controllability is the minimum distance from the origin to the boundary of this region. Calculation of the boundary of the recovery region is a difficult problem which requires, in most cases, the calculation of many quadratures. The overriding objection to this method is the complication involved in the multiple control case. Another objection is that Viswanathan chooses to bound control magnitude and does not attempt to perform any sort of minimization with respect to quantity of control used, citing bounded control magnitude as the more realistic situation. It is usually the case, however, that quantity of control (e.g., fuel in thruster, stored angular momentum in control moment gyro, CMG) is the primary consideration, not saturation of the controller. In later writings, these authors suggest a variety of alternatives to this definition of the recovery region including one based on minimum energy control which is the approach selected in this paper.

The idea of a "figure of merit" for controllability based on minimum energy control had previously been introduced by Kalman et al.[5] We will also use minimum energy control because of its convenient calculation, but add to what has appeared in the literature a transformation of the result (steps 3 and 4 of the following section) which we believe is essential to the practical interpretation of the controllability measure.

Dynamic Measure of Controllability

The measure of controllability formulated here utilizes some of the characteristics of these methods. It results from a four-step procedure:

1) Find the minimum control energy strategy for driving the system from a given initial state to the origin in the prescribed time. ["Control energy" is defined as

$$E = \frac{1}{2}\int_0^T u^T R u \, dt$$

where R is a positive definite weighting matrix.]

2) Find the region of initial states which can be driven to the origin with constrained control energy and time using the optimal control strategy. This region is bounded by an ellipsoidal surface in state space.

3) Scale the axes so that a unit displacement in every direction is equally important to control.

4) The degree of controllability is a linear measure of the weighted "volume" of the ellipsoid in this equicontrol space.

Step 1: Step 1 can be stated mathematically as follows:

$$\min E = \frac{1}{2}\int_0^T u^T R u \, dt$$

subject to

$$\dot{x} = Ax + Bu, \qquad x(0) = x_0, \qquad x(T) = 0 \tag{1}$$

The Hamiltonian for this problem is

$$H = \tfrac{1}{2} u^T R u + p^T(Ax + Bu)$$

so that

$$\dot{p} = -A^T p \qquad p(0),\ p(T) \text{ free} \tag{2}$$

$$u^*(t) = -R^{-1} B^T p(t) \tag{3}$$

where $u^*(t)$ is the optimal control. It is possible to solve explicitly for $p(t)$ and $u^*(t)$, but the above relations are the only results we will require.

Step 2: In order to carry out step 2 of the procedure, we will require an expression for the optimum cost

$$E^* = \frac{1}{2}\int_0^T u^{*T} R u^* \, dt$$

as a function of the initial state. To this end, we seek a relation of the form

$$x = Vp \tag{4}$$

since p is a function of the initial state. Differentiating Eq. (4), substituting Eqs. (1-3), and noting that the resulting equation set equal to zero must hold for arbitrary p, we find that

$$\dot{V} = AV + VA^T - BR^{-1}B^T \tag{5}$$

with the boundary condition

$$V(T) = 0 \tag{6}$$

to satisfy the requirement that $x(T) = 0$ since in general $p(T)$ is not zero. We choose this boundary condition for V as a matter of convenience; any other terminal value which satisfies the requirement $V(T)p(T) = 0$ would produce the same result for the control energy. The reason for not using the usual relation $p = Wx$ is that in order for $p(T)$ not to be zero, $W(t)$ would have to be poorly defined at $t = T$.

Corresponding to the usual cost expression

$$J = \tfrac{1}{2} x(0)^T W(0) x(0)$$

we expect the energy cost to have the inverse form

$$E = \tfrac{1}{2} x(0)^T V(0)^{-1} x(0) \tag{7}$$

The validity of this expression can be verified by generalizing the initial time to t_0 and differentiating E with respect to it.

Equation (7), with a fixed value of E, defines an n-dimensional ellipsoidal surface in initial state space. Any point within the ellipsoid can be returned to the origin in time T with energy E using the optimal control in Eq. (3). Although the energy expression (7) is simpler than that appearing in Eq. (1), the differential equation for V in Eq. (5) remains to be solved. An analytic solution applicable to the dynamics of flexible space structures is given in Ref. 6.

Step 3 is to scale the axes so that a unit displacement in every direction is equally important. "Importance" in this context should not be related to the accuracy with which a variable is ultimately controlled, but rather to the size of the controlled excursion one would like to be able to achieve. Scaling should make a more important variable smaller in the scaled space so as to emphasize the need to improve the control over that variable.

To this end, let $x_{i_{\min}}$ be the minimum state excursion one would like to be able to return to the origin in a given time using a prescribed control energy. Then define the transformation

$$z = Dx$$

where

$$D = \begin{bmatrix} \frac{1}{|x_{1_{\min}}|} & & \\ & \ddots & \\ & & \frac{1}{|x_{n_{\min}}|} \end{bmatrix} \tag{8}$$

so that unit values of z in any direction represent controllable displacements of equal importance. If controlling a given state is deemed less important (which is useful to recognize since it requires less control capability), the corresponding state in z space is made larger.

Consider a two-dimensional case in which it is as important to control an initial displacement in the x_1 direction twice as large as one in the x_2 direction. In this case, the ellipsoid defined by Eq. (7) is an ellipse in x space. Let the ellipse have the shape illustrated in Fig. 1a. This represents the ideal allocation of control since we are able to control a maximum displacement in the x_1 direction exactly twice as large as one in the x_2 direction. Figure 1b illustrates that the ellipse becomes a circle when transformed to equicontrol space via Eq. (8).

Step 4 is to measure the controllability represented by the ellipsoid in equicontrol space (z space). From the preceding discussion we see that any deviation from a circle in equicontrol space represents a less than ideal control allocation.

After considering a number of alternatives, the degree of controllability was chosen to be the following:

$$\mathrm{DC} = \left[V_S + \frac{V_S}{V_E}(V_E - V_S)\right]^{1/n} \tag{9}$$

where V_E is the n-dimensional volume of the ellipsoid in equicontrol space and V_S is the volume of the largest inscribed sphere; n is the dimension of the state space. The first term on the right-hand side of Eq. (9) is the predominant term in the controllability measure; it reflects the smallest magnitude of initial state in equicontrol space which can be driven to the origin in the specified time using the specified control energy. If the controls were ideally allocated, the initial condition surface would be a sphere and V_S would be the controllability measure. The second term in Eq. (9) adds a smaller amount to degree of controllability (DC) to recognize the larger region of state space from which the system can recover if the surface is not spherical. The additional volume, $V_E - V_S$, is scaled by V_S/V_E so that the most this term can add, as $V_E \to \infty$, is V_S. The nth root of the weighted volume is taken as the controllability measure to make it proportional to the linear dimensions of the region from which the system can recover. The volume weighting scheme for a two-dimensional case (volumes are areas) is depicted in Figs. 2a-c.

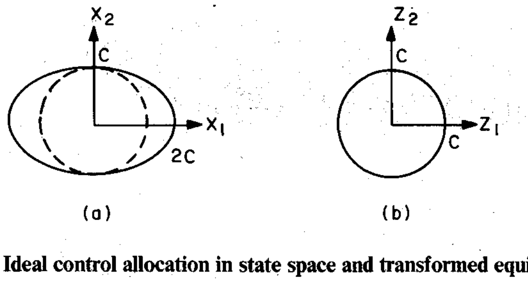

Fig. 1 Ideal control allocation in state space and transformed equicontrol space.

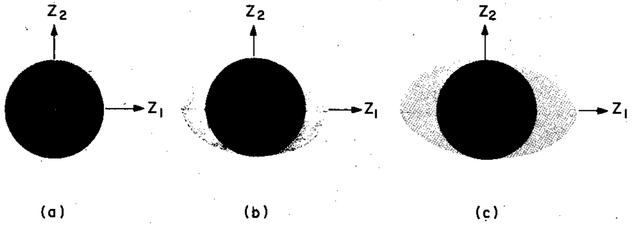

Fig. 2 Control weighting scheme for computing degree of controllability, a) ideal control distribution, b) slightly distorted distribution, c) very distorted distribution. Shading indicates relative weighting.

What remains to be shown are the mechanics of computing the n-dimensional volumes V_S and V_E. The volume enclosed by the quadratic surface $x^TAx = d$, with A positive definite and symmetric, is proportional to the product of the reciprocals of the square roots of the eigenvalues of A. Since volume has little intuitive significance in spaces of higher dimension than 3, we take the "volume" to be simply

$$V = \left(\prod_{i=1}^{n} \sqrt{\lambda_i}\right)^{-1} \tag{10}$$

To apply this result to the case at hand, first substitute Eq. (8) into Eq. (7) to obtain the equation of the ellipsoidal surface in equicontrol space

$$E = \tfrac{1}{2} z_0^T (DV_0D)^{-1} z_0 \tag{11}$$

V_E is then given by Eq. (10) where λ_i are the eigenvalues of $(DV_0D)^{-1}$. More conveniently, if the ν_i denote the eigenvalues of DV_0D, the ellipsoidal volume is also given by

$$V_E = \prod_{i=1}^{n} \sqrt{\nu_i} \tag{12}$$

and the spherical volume is the shortest distance to the surface, $1/\sqrt{\lambda_{\max}}$, to the nth power, or alternatively,

$$V_S = \left(\sqrt{\nu_{\min}}\right)^n \tag{13}$$

The degree of controllability then can be computed using Eqs. (9), (12), and (13) and actually becomes zero when the system is uncontrollable; the ellipsoid collapses to zero in the uncontrollable direction so that $\nu_{\min}$ is zero.

One further consideration is important in defining the degree of controllability of a system—how the measure varies with number of actuators. The degree of controllability has been defined in terms of a constraint on control energy with no reference to a constraint on control magnitude. But it seems appropriate to recognize the fact that a system with more actuators has greater control capability when there is a limit on control magnitude—as is always the case. The measure of controllability as defined above can be made to vary directly with the number of actuators, if they have equal effectiveness, by scaling the elements of R inversely with m—the number of actuators in the system. Usually R is taken diagonal, and if the diagonal elements $R_{0_{ii}}$ are first chosen to reflect the relative cost of using the different actuators, then the final elements of R are defined to be

$$R_{ii} = R_{0_{ii}}/m \tag{14}$$

where m is the total number of actuators.

Dynamic Measure of Observability

Any measure of the observability of a dynamic system should reflect the amount of information which can be derived about the system states from the sensor outputs in a given amount of time as directly as possible. The means of obtaining this information is by associating with the system an observer whose states, $\hat{x}$, are estimates of the true states of the system. The more information that is obtained about the system, the smaller the estimation error becomes.

In the parallel problem of indicating the controllability of a system with a given set of actuators it is not obvious how the control should be used so as to realize the best possible control of the system. What is "best" might be defined in a number of reasonable ways. But in this case, if we use a linearized description of the system dynamics, then the best way to process the sensor data so as to minimize estimation error is obvious. The linear estimator which minimizes the state estimation error vector, $e = \hat{x} - x$, in a mean-square sense, i.e., minimizes

$$S = \overline{e^T M e} \tag{15}$$

where M is some weighting matrix, is the Kalman filter.

With the state dynamics given in Eq. (1) and the measurement relation written as

$$y = Cx + v \tag{16}$$

the estimation error covariance matrix with Kalman filtering obeys the equation

$$\dot{P} = AP + PA^T - PC^T N^{-1} CP + Q \tag{17}$$

where P is the estimation error covariance matrix, and N and Q are the measurement and driving noise intensity matrices, respectively. But the estimation error, or its covariance matrix, is an inverse indication of the amount of information one has about the state. We wish the measure of observability to be a direct indicator of information, so to that end we introduce the information matrix—the inverse of the error covariance matrix.

$$J = P^{-1} \tag{18}$$

One other consideration is that the observability measure should be a property of the system and a set of sensors—nothing else. Since the measurement noise is a property of the set of sensors being evaluated, we retain its inclusion in Eq. (17) in the form of N but do not include the effect of state driving noise, because that is an external influence not related to the sensor set. Thus, if we set $Q = 0$, then Eq. (17) in terms of J becomes

$$\dot{J} = -JA - A^T J + C^T N^{-1} C \tag{19}$$

Take as the standard situation the case in which there is no information about the state initially and data is collected up to a specified time T. Then $J(0) = 0$ and one is interested in $J(T)$.

Having the information matrix at time T, we need some scalar measure of how large the matrix is as an indication of how much information has been generated by optimal processing of the sensor data. One way of measuring the size of $J(T)$ is by reference to the quadratic surface

$$v^T J^{-1} v = 1 \tag{20}$$

Equation (20) defines an ellipsoidal surface in v space. The larger J is the larger the volume encompassed by the surface in Eq. (20), so that the volume is related directly to the amount of information obtained about the system.

Typically, however, some components of x will be of greater concern than others, especially considering that different units of measurement will apply to different components. Regardless of how one were to approach the definition of a measure of observability, it is clear that a scaling of the state variables to reflect their relative importance to the success of the mission would be required. To that end, define the transformation

$$w = Fv$$

$$F = \begin{bmatrix} |e_{1_{\max}}| & & \\ & \ddots & \\ & & |e_{n_{\max}}| \end{bmatrix} \tag{21}$$

where $e_{i_{\max}}$ are the maximum errors one is willing to tolerate in the directions x_i. The more error one is willing to tolerate in a given direction, the greater the transformed state becomes in that direction. Thus the scaling is consistent with the ideas presented in the last section. Also note that v has units of reciprocal error, so w is dimensionless as was z in the control case.

Now that the axes have been scaled so that it is equally important to obtain information in each direction, one can use the same definition for the degree of observability as was used for controllability when applied to equicontrol space. Again, the ideal sensor distribution would produce a sphere in w space, so that the degree of observability (DO) involves a spherical volume plus a lesser weighted excess volume due to the nonideality of the distribution. Specifically,

$$\mathrm{DO} = \left[V_S + \frac{V_S}{V_E}(V_E - V_S) \right]^{1/n} \tag{22}$$

with

$$V_E = \prod_{i=1}^{n} \sqrt{\nu_i}, \qquad V_S = \left(\sqrt{\nu_{\min}}\right)^n$$

and the ν_i are the eigenvalues of $FJ(T)F^T$.

The remaining problem is to solve the differential equation (19) for J so as to write out explicitly $J(T)$. This equation and its boundary condition are similar to those for V, Eqs. (5) and (6), required in the calculation of the degree of controllability. Define a backward time variable, $\tau = T - t$, so that $\mathrm{d}J/\mathrm{d}\tau = -\mathrm{d}J/\mathrm{d}t$. Then in terms of τ, Eq. (19) becomes

$$\dot{J} = JA + A^T J - C^T N^{-1} C, \qquad J(T) = 0 \tag{23}$$

This is the same as the equation and boundary condition for V with the following substitutions. V equation: A, B, R; J equation: A^T, C^T, N. So if a subroutine is prepared to produce $V(0)$ given A, B, and R, that same subroutine can be used to produce $J(T)$ by use of the substitutions indicated.

It can be seen from the relations above that V and J are formally related as duals. However, it is felt that the physical interpretations of these definitions of controllability and observability would be lost if one were to define just one and then simply appeal to duality to obtain the other.

Recognition of Component Failures

Because of the realistic possibility of components failing during the operating lifetime of the system, one would like the degree of controllability (and observability) to be averaged in some way over the set of component failure combinations which the system may experience. To this end, let f be an indicator of the state of failures of the components, and let the vector $\boldsymbol{l}$ represent their locations. Then, for a given set of operating actuators, one can compute the degree of controllability, $\mathrm{DC}(\boldsymbol{l}, f)$, using the method described previously.

The component locations indicated by $\boldsymbol{l}$ are deterministic; they will be adjusted subsequently to optimize the degree of controllability. But f is a random variable with a time-dependent probability distribution. Thus $\mathrm{DC}(\boldsymbol{l}, f)$ is also a random variable with a time-dependent probability distribution defined by the distribution of f. To define a meaningful deterministic performance measure, one would logically use the expected value of $\mathrm{DC}(\boldsymbol{l}, f)$ with the expectation taken over the distribution of f, the failure state for the system components. This yields a performance measure which depends on time, t. It represents a measure of the expected performance of the system at time t in view of the probabilities of the various failure states at that time.

But this control system is required to operate over a certain period T_m which might represent the time between maintenance visits. Rather than optimize the degree of controllability at any one time, such as the end of that period, it would seem more meaningful to optimize the average degree of controllability over the whole period. In this average, the performance resulting from failure states which are likely over longer periods would be weighted more heavily than those likely to exist over shorter periods. And a probability weighted measure of performance over the whole operating period is obtained rather than just a measure of performance at one time.

The average of the expected degree of controllability over the mission period T_m is taken as the final measure.

$$\mathrm{DC}_{\mathrm{ave}}(l) = \frac{1}{T_m}\int_0^{T_m} \overline{\mathrm{DC}}(l,f)\,\mathrm{d}t \tag{24}$$

But the expected DC is simply a weighted sum over the different failure states,

$$\overline{\mathrm{DC}}(l,f) = \sum_i \mathrm{DC}(l,f_i)\,P_i(t) \tag{25}$$

where $P_i(t)$ is the probability of failure state f_i at time t. The final measure can be expressed as

$$\mathrm{DC}_{\mathrm{ave}}(l) = \sum_i \mathrm{DC}(l,f_i)\frac{1}{T_m}\int_0^{T_m} P_i(t)\,\mathrm{d}t \tag{26}$$

and depends on T_m and the component failure statistics as well as the locations. The modified degree of observability is computed in the same way. A general expression for the average of $P_i(t)$ as required in this relation is given in Ref. 7 for the usual assumption of independent failures and exponential distribution of time to failure.

Optimum Component Placement

Having a computable measure of how well the structure can be controlled (observed) with any given set of actuators (sensors), with the expected effect of component failures throughout the mission reflected in the measure, one can then seek to optimize the choice of component locations, for a given number, so as to maximize the performance measure. This task may be computationally burdensome when dealing with a large number of components, but it is conceptually straightforward.

A constraint which will likely apply in most applications is that component placement will be restricted to a discrete set of permissible locations. Structural considerations, for example, may require that control moment gyros be mounted only at the joints of a truss structure. If this is true for all of the components, then the placement optimization problem is in the nature of an integer programming problem. Many algorithms have been described in the literature for solving integer programming problems; nothing has been added to that art in this work. The examples which follow are intended only to illustrate the nature of this step. They were restricted to a small number of components and optimization was accomplished by global search—by testing all admissible combinations of component locations.

Choice of Component Number

Having the optimum set of component locations and the corresponding maximum degree of controllability (observability) for a given number of components, one can compute this maximum performance measure for several choices of component number. The choice of how many actuators and sensors to use in the system cannot be resolved as an optimization problem unless additional factors are incorporated in the criterion. The degree of controllability or observability will always improve with additional components if the best locations are used in each case.

However, it should be informative to observe the trend of the performance measure with number of components. Some locations are more advantageous than others, such as the placement of torque actuators near the nodes of important modes. With the realistic restriction that only one component can be placed at any one of the allowable locations, one should expect to see diminishing returns in performance with increasing number as the more favorable locations are occupied. This information should be helpful to the designer in making the tradeoff between improved performance and increased cost, power required, etc.

Application to Beam

To illustrate the methodology defined above, actuator and sensor placement and number were considered for the case of a free–free beam. The beam was modeled as in Ref. 6 with the states representing the modal amplitudes and rates of the first three flex modes; force actuators were used for control (control period was 10 s). In all trials the amplitude rate states were scaled by the factor $1/\omega_i$ relative to the amplitude states where ω_i is the corresponding modal frequency. The actuators were assumed equally efficient ($R_0 = I$), but the elements were scaled by (1/No. of actuators) to reflect saturation of the controllers. This scaling was chosen to produce a result which is proportional to the number of actuators of equal effectiveness. For example, two actuators at the same location have a degree of controllability twice that of a single actuator at that position.

The effect of actuator location on the degree of controllability of a three-mode representation of a uniform free–free beam is shown in Fig. 3. This figure is a plot of DC as a function of the location of a single force actuator along the length of the beam. As an aid to interpretation of these results, the mode shapes for the three simulated modes are given in Fig. 4. No failures are considered. As one would expect, DC is zero at each node of the three modes because, with just one actuator, one mode is uncontrollable in those cases—and the uncontrollability of any mode is reflected in a zero DC. The degree of controllability rises to intermediate peaks between the nodal points and has its maximum at the ends of the beam where the modal deflections of all three modes are greatest. For this system, then, it is clear that the

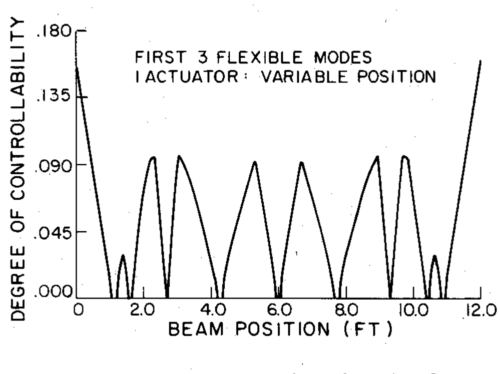

Fig. 3 Degree of controllability for a free–free beam.

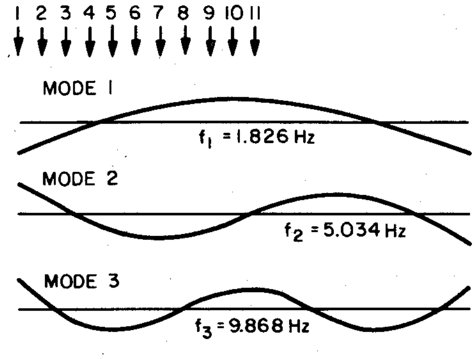

Fig. 4 Modeled mode shapes and component test positions.

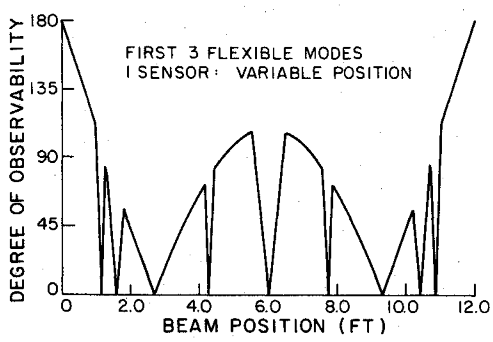

Fig. 5 Degree of observability for a free–free beam.

Table 1 Optimal degrees of controllability and locations for varying actuator number

Actuator No.	Location[a]		Degree of controllability	
	No fail	Fail	No fail	Fail
1	1	1	0.1609	0.1017
2	11,1	5,1	0.2791	0.1657
3	5,11,1	10,5,1	0.3856	0.2305
4	6,5,11,1	6,11,5,1	0.4879	0.2980

[a]Location number refers to test position from the end of the beam (actuators were restricted to one side of beam only).

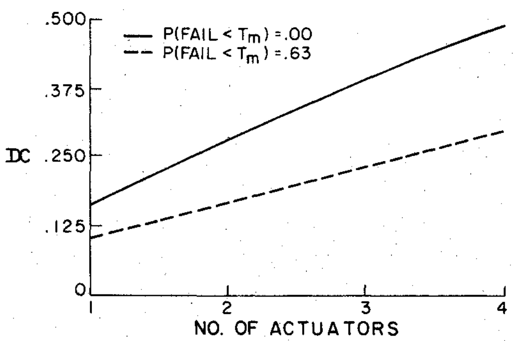

Fig. 6 Optimal degree of controllability vs number of actuators for three-mode simulation.

end of the beam is the optimum location for one actuator no matter how many are used.

The corresponding results for the degree of observability are quite similar as one would expect. Figure 5 shows DO as a function of the location of a translation rate sensor along the beam which is again represented by the dynamics of its first three flexible modes. The amplitude rate states were scaled by the factor ω_i relative to the amplitude states where ω_i is the corresponding modal frequency. The sensors were assumed equally noisy ($N=I$).

Optimum actuator locations for this system were found for 1, 2, 3, and 4 actuators with and without component failures considered. As one might expect, it will usually be true that the best places to locate control system components are the same with and without consideration of possible component failures. But as this example illustrates, this is not always the case. Permissible actuator locations were restricted to the 11 discrete locations indicated in Fig. 4. Only half of the beam was searched for favorable locations because of the symmetry of all the modes. The component mean time to failure was taken equal to the mission time, so the probability that any one actuator fails before the end of the mission is 0.63, and the average probability of any one actuator failure over the mission period is 0.37. All calculations were performed with a computer program given in Ref. 6.

The detailed results are given in Table 1. For a single actuator, the optimal location with and without consideration of failures is in position 1 at the end of the beam as was anticipated above. For the case of two actuators, positions 1 and 11 (end and center) are best for no failures and positions 1 and 5 are optimal when failures are considered. The reason for this difference can be seen by examining Fig. 3 which illustrates degree of controllability vs actuator position for a single actuator along the three-mode beam. The DC at the center of the beam (11) is zero because that is the location of a node of the second mode. However, the center is also an antinode of the first and third modes (see Fig. 4), so that as long as some control is maintained over the second mode by another actuator, the center is an excellent location for a secondary actuator. Thus positions 11 and 1 are optimal locations for two actuators and positions 5, 11, 1 are optimal for three. But once the possibility of an actuator failure is introduced, the penalty for losing an actuator at 1 or 5 and being left with only the one at 11, which leaves the second mode uncontrollable, weighs heavily into the average DC shifting the optimal location away from the center.

Finally, the effect of the number of actuators on the degree of controllability of the three-mode representation of the beam is shown in Fig. 6 both with and without failures considered. This is a plot of DC data appearing in Table 1; each value is the degree of controllability resulting from optimal placement of the corresponding number of actuators. Both curves are seen to be essentially linear over the range of actuator numbers shown. The reason for this is clear when one notes the DC as a function of the location of a single actuator shown in Fig. 3; after locating the first actuator in position 1, there are several possible positions for the next few actuators which have almost equal effectiveness. If Fig. 6 were to be extended to larger numbers of actuators, it would show the expected diminishing returns as the more favorable positions become occupied.

Conclusions

A methodology has been presented which is intended to assist the designer of a control system for a large space structure to decide how many sensors and actuators should be incorporated in the system and where they should be placed on the structure. This approach is intended to be especially useful in the early stages of the evolution of the system, before a complete control system concept has been defined. This methodology uses quantitative measures of the controllability and observability of the system for given sets of actuators and sensors. The effect of possible component failures during the mission period was incorporated in the measures. The question of actuator and sensor placement is then resolved by finding the locations which maximize these performance measures. The number of components to use cannot be determined by optimizing these measures because the controllability and observability always improve with increased number of components if they are optimally located. However, the degree of the improvement in these measures with component number can be determined, and this information can be used along with data on cost, power required, etc., to decide how many components to use.

These procedures were illustrated for the case of control of a uniform free–free beam. Optimal actuator locations were found and the variation of maximum degree of controllability with number of actuators was determined for up to four actuators. Cases were shown in which the recognition of possible actuator failures resulted in significantly different optimum actuator locations than without consideration of failures. The results are intuitively clear when dealing with a simple beam, but it is hoped that this methodology will be useful in more realistically complicated design situations by providing a rational quantitative basis for addressing the questions of control system actuator and sensor number and placement.

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References

[1]Horner, G.C., "Optimum Damper Locations for a Free-Free Beam," *2nd Large Space Systems Technology Review*, NASA CP 2169, Nov. 1980.

[2]Hughes, P.C. and Skelton, R.E., "Controllability and Observability for Flexible Spacecraft," *Journal of Guidance and Control*, Vol. 3, Sept.-Oct. 1980, pp. 452-459.

[3]Juang, J.N. and Rodriguez, G., "Formulations and Applications of Large Structure Actuator and Sensor Placement," *Proceedings of 2nd VPI & SU/AIAA Symposium (Dynamics and Control of Flexible Spacecraft)*, Blacksburg, Va., June 1979, pp. 247-261.

[4]Viswanathan, C.N., Longman, R.W., Likins, P.W., "A Definition of the Degree of Controllability—A Criterion for Actuator Placement," *Proceedings of 2nd VPI & SU/AIAA Symposium*, Blacksburg, Va., June 1979, pp. 369-384.

[5]Kalman, R.E., Ho, Y.C., and Narendra, K.S., "Controllability of Linear Dynamical Systems," *Contributions to Differential Equations*, Vol. 1, No. 2, 1963, pp. 189-213.

[6]Vander Velde, W.E. and Carignan, C.R., "A Dynamic Measure of Controllability and Observability for the Placement of Actuators and Sensors on Large Space Structures," MIT Space Systems Lab, Massachusetts Institute of Technology, Cambridge, Mass., Rept. 2-82, Jan. 1982.

[7]Carignan, C.R. and Vander Velde, W.E., "Number and Placement of Control System Components Considering Possible Failures," MIT Space Systems Lab, Massachusetts Institute of Technology, Cambridge, Mass., Rept. 5-82, March 1982.